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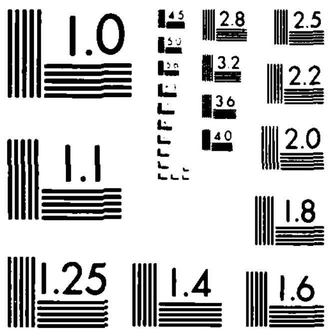

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PERIODIC SOLUTIONS OF PRESCRIBED ENERGY FOR A CLASS OF HAMILTONIAN SYSTEMS

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ABSTRACT

The main result of this paper is the following theorem: Let $p,q \in \mathbb{R}^n$, $H = H(p,q) \in C^2(\mathbb{R}^{2n},\mathbb{R})$ and let $H^{-1}(1)$ be the boundary of a compact neighborhood of 0 with $\nabla H \neq 0$ on $H^{-1}(1)$. If further $p \cdot H_p > 0$ on $H^{-1}(1)$ when $p \neq 0$, then the Hamiltonian system of ordinary differential equations

$$\dot{p} = -H_q(p,q), \quad \dot{q} = H_p(p,q)$$

possesses a periodic solution on $H^{-1}(1)$. The proof involves minimax arguments from the calculus of variations.

AMS (MOS) Subject Classifications: 34C15, 34C25

Key Words: Hamiltonian system, periodic solutions, minimax

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SIGNIFICANCE AND EXPLANATION

Hamiltonian systems of ordinary differential equations model the motion of a discrete mechanical system when no frictional forces are present. A basic property of such systems is that "energy" is conserved. Therefore solutions of Hamiltonian systems lie on surfaces of fixed energy. The main result of this paper is a fairly general criterion for such a surface to possess a periodic solution.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

PERIODIC SOLUTIONS OF PRESCRIBED ENERGY FOR A CLASS OF HAMILTONIAN SYSTEMS

Vieri Benci* and Paul H. Rabinowitz**

Introduction

Let $p,q \in \mathbb{R}^n$ and $H = H(p,q) : \mathbb{R}^{2n} \to \mathbb{R}$ be smooth. The problem to be studied here is the existence of periodic solutions of the associated Hamiltonian system of ordinary differential equations

(HS) $$\dot{p} = -\frac{\partial H}{\partial q}(p,q), \quad \dot{q} = \frac{\partial H}{\partial p}(p,q)$$

where $\cdot \equiv \frac{d}{dt}$. Setting $z = (p,q)$, (HS) can also be written more succinctly as

$$\dot{z} = \mathcal{J} H_z(z)$$

where $\mathcal{J} = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$ and I is the n dimensional identity matrix. As is well known any solucion $z(t)$ of (HS) satisfies $H(z(t)) \equiv$ constant, i.e. the "energy" H is an integral of the motion. Normalizing this constant to be 1, set $\mathcal{D} \equiv H^{-1}(1)$. For $\xi,\eta \in \mathbb{R}^j$, $\xi \cdot \eta$ will denote their inner product. Our main result is

Theorem 1: Suppose H satisfies

(H_1) $H \in C^2(\mathbb{R}^{2n},\mathbb{R})$,

(H_2) $\mathcal{D}$ is the boundary of a compact neighborhood of 0 and $H_z \neq 0$ on $\mathcal{D}$ (i.e. $\mathcal{D}$ is a manifold).

(H_3) $p \cdot H_p \neq 0$ if $p \neq 0$.

Then (HS) possesses a periodic solution on $\mathcal{D}$.

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Theorem 1 has several predecessors. Seifert [1] considered Hamiltonians of the form

$$H(p,q) = \sum_{i,j=1}^{n} a_{ij}(q) p_i p_j + V(q) ,$$

i.e. H is a sum of kinetic and potential energy terms where $\mathcal{B} \equiv \{q \in \mathbb{R}^n | V(q) < 1\}$ is diffeomorphic to the closed unit ball in $\mathbb{R}^n$ and $\partial\mathcal{B}$ is a manifold, the matrix $(a_{ij}(q))$ is uniformly positive definite in $\mathcal{B}$, and H is smooth. Using geodesic arguments from geometry, Seifert proved there exists a periodic solution of (HS) of a special type on $\mathcal{D}$. Generalizing his arguments, Weinstein [2] permitted a more general kinetic energy term $K(p,q)$ where for fixed q, K is even and convex in p while Gluck and Ziller [3] relaxed the condition on $\mathcal{B}$ merely requiring $\mathcal{B}$ to be compact with its boundary a manifold. See also Hayashi [4] and Benci [5] for results related to [3]. Another approach was made to (HS) in Rabinowitz [6] for $H = K + V$ where V satisfied Seifert's condition and $p \cdot K_p > 0$ for $p \neq 0$. A case not covered by Theorem 1 but which can be obtained by similar but simpler arguments was given in [7] in which $\mathcal{D}$ is the boundary of a compact star-shaped neighborhood of 0.

In a different direction from Theorem 1, there has been some recent work on the multiplicity of solutions of (HS) on $\mathcal{D}$, generally when $\mathcal{D}$ bounds a convex region in $\mathbb{R}^{2n}$. See e.g. Ekeland-Lasry [8], Ambrosetti-Mancini [9], van Groesen [10], Berestycki-Lasry-Mancini-Ruf [11], and Ekeland [12].

We will prove Theorem 1 by a direct variational approach using minimax arguments. The proof relies in part on ideas from [5-6]. Let $z(t) = (p(t),q(t))$ be 2π periodic and

$$A(z) \equiv \int_0^{2\pi} p \cdot \dot{q} dt .$$

[illegible] of (HS) will be obtained as a critical point of A restricted to $\mathcal{M} \equiv \Psi^{-1}(1)$

$$\Psi(z) \equiv \frac{1}{2\pi}\int_0^{2\pi} \bar{H}(z)dt ,$$

$\bar{H}(z) = H(z)$ on $\mathcal{D}$ and is suitably modified on $\mathbb{R}^{2n}\setminus\mathcal{D}$. This critical point is produced as a minimax of $A|_M$ over an appropriate class of subsets of M. In this approach, the unknown period appears as a Lagrange multiplier.

The modified Hamiltonian $\bar{H}$ will be defined in §1 where some simple corollaries of Theorem 1 will also be obtained. In §2 the functional analytical framework in which the problem is treated is introduced. The properties of M and $A|_M$ such as the Palais-Smale condition are dealt with in §3. Theorem 1 is proved in §4. A dual variational argument is used in §5 to give an alternate approach to Theorem 1. In §6 a priori bounds from above and below are obtained for the unknown period of any solution of (HS) in terms of $A(z)$. Lastly in §7, the results of §4 and 6 are used to prove a stronger version of Theorem 1 with (H_1) replaced by

(H_1') $$H \in C^1(\mathbb{R}^{2n},\mathbb{R}) .$$

An intriguing open question concerning (HS) is whether Theorem 1 remains true or is false if hypothesis (H_3) is omitted.

§1. The Modified Hamiltonian

For technical reasons that will become clear later, the Hamiltonian will be redefined outside of a neighborhood of $\mathcal{D}$. Suppose H satisfies (H_1)-(H_3). Then $H(0) < 1$ and $H > 1$ outside of the neighborhood of 0 bounded by $\mathcal{D}$. Without loss of generality we can assume $H(0) < \frac{1}{2}$. Our initial modification of H will allow us to assume $H > 0$, is a multiple of $|z|^2$ near 0, satisfies (H_1)-(H_3), and H_{zz} is uniformly bounded.

Indeed since $H(0) < \frac{1}{2}$, $\rho > 0$ can be chosen so that $\rho|z|^2 > \frac{3}{2}$ if $H(z) > 1$ and $\rho^{-1}|z|^2 < \frac{1}{4}$ if $H(z) \leq 1$. Let $\chi \in C^\infty(\mathbf{R},\mathbf{R})$ such that $\chi(s) = 0$ if $s \leq \frac{1}{4}$; $\chi(s) = 1$ if $s \geq \frac{1}{2}$, and $\chi'(s) > 0$ if $s \in (\frac{1}{4}, \frac{1}{2})$. Define

$$\tilde{H}(z) = \chi(H(z) - 1)[\rho|z|^2 - H(z)] + H(z) + \chi(1 - H(z))[\rho^{-1}|z|^2 - H(z)] .$$

Then $\tilde{H} \in C^2(\mathbf{R}^{2n},\mathbf{R})$, $\tilde{H} = \rho^{-1}|z|^2$ near $z = 0$, and $\tilde{H} = H$ near $\mathcal{D}$. Moreover if $\tilde{H}(z) = 1$, then $z \in \mathcal{D}$. To see this, suppose $H(z) > 1$. Then

$$\tilde{H}(z) = \chi(H(z) - 1)[\rho|z|^2 - H(z)] + H(z) .$$

If $H(z) \geq \frac{3}{2}$, $\tilde{H}(z) = \rho|z|^2 > \frac{3}{2} > 1$ while if $H(z) \in (1, \frac{3}{2})$, $\tilde{H}(z) \geq H(z) > 1$. Similar reasoning shows $\tilde{H}(z) < 1$ if $H(z) < 1$. Thus $\tilde{H}^{-1}(1) = \mathcal{D}$. A related argument shows $\tilde{H}(z) > 0$ if $z \neq 0$. It is clear that $\tilde{H}$ satisfies (H_1)-(H_2). To verify (H_3), by the definition of χ, it suffices to show that for $p \neq 0$,

$$\begin{aligned}
(1.1) \qquad p \cdot \tilde{H}_p(z) = {} & \{\chi'(H(z) - 1)[\rho|z|^2 - H(z)] - \chi'(1 - H(z))[\rho^{-1}|z|^2 - H(z)]\}p \cdot H_p(z) \\
& + (1 - \chi(H(z) - 1) - \chi(1 - H(z)))p \cdot H_p(z) \\
& + 2|p|^2[\chi(H(z) - 1)\rho + \chi(1 - H(z))\rho^{-1}] > 0
\end{aligned}$$

if $z \in H^{-1}[\frac{1}{2}, \frac{3}{2}]$. Again this follows from our choice of ρ and (H_3).

Since $\mathcal{D}$ is compact, there is a $\beta > 0$ such that $|z| \leq \beta$ for $z \in \mathcal{D}$. Let $\hat{\chi} \in C^\infty(\mathbf{R},\mathbf{R})$ such that $\hat{\chi}(s) = 1$ for $s \leq 2\beta$, $\hat{\chi}(s) = 0$ for $s \geq 4\beta$, and $\hat{\chi}'(s) < 0$ for $s \in (2\beta,4\beta)$. Set

$$\hat{H}(z) = \hat{\chi}(|z|)\tilde{H}(z) + (1 - \hat{\chi}(|z|))\hat{\rho}|z|^2 .$$

Then it is easy to check that for $\hat{\rho}$ chosen so that $\hat{\rho}|z|^2 > \tilde{H}(z)$ for $|z| \in (2\beta,4\beta)$, $\hat{H}$

possesses the properties verified above for $\tilde{H}$, $\hat{H}$ is a multiple of $|z|^2$ for large $|z|$, and satisfies

(H_4) $H_{zz}(z)$ is uniformly bounded.

Remark 1.2: The above arguments work equally well if H merely satisfies (H_1) and (H_3) in a neighborhood of $\mathcal{D}$.

Next hypothesis (H_3) will be used to decompose H into a sum of kinetic and potential energy terms. Set $U(q) = \hat{H}(0,q)$ and $K(p,q) \equiv K(z) \equiv \hat{H}(z) - U(q)$. Note that $K(z) > 0$ via (H_3) and $K, U \in C^2$ via (H_1). Moreover

Proposition 1.3: K satisfies the following properties:

(K_1) $K(0,q) = 0$

(K_2) $p \cdot K_p(z) > 0$ if $p \neq 0$

(K_3) $|K_p(z)| \leq a_1(1 + |z|)$

(K_4) $K(z) \leq a_1(1 + |z|)|p|$

(K_5) $|K_q(z)| \leq a_2|p|$

(K_6) $K_{zz}(z)$ is uniformly bounded in $\mathbb{R}^{2n}$.

(In (K_3)-(K_5) and later, a_i denotes a constant.)

Proof: (K_1) and (K_2) follow from the definition of K and (H_3), and (K_3), (K_6) from (H_4). Since

$$K(p,q) = \int_0^1 \frac{d}{ds} K(sp,q)ds = \int_0^1 p \cdot K_p(sp,q)ds \ , \tag{1.4}$$

(K_3) and (1.4) imply (K_4). Similarly

$$K_q(p,q) = H_q(p,q) - H_q(0,q) = \int_0^1 H_{pq}(sp,q)p\ ds \tag{1.5}$$

so (H_4) and (1.5) give (K_5).

To define $\bar{H}$, one final modification of H is required. Let

$$\Omega_s = \{q \in \mathbb{R}^n | U(q) < s\} \ .$$

By (H_2), $U_q \neq 0$ on $\partial\Omega_1$ and there exist constants $d, \beta > 0$ such that $U_q(q) \neq 0$ and

$$|U_q(q)| \geq \beta U(q) \quad \text{if} \quad q \in \Omega_{1+2d} \setminus \Omega_{1-2d} . \tag{1.6}$$

Let $\phi \in C^2$ be defined for $s < 1 + 2d$ such that

(ϕ_1) $\phi(s) = s$, $s \leq 1 + d$

(ϕ_2) $\phi'(s) \geq 1$, $s < 1 + 2d$

(ϕ_3) $\phi(s) = (s - (1 + 2d))^{-2}$, s near $1 + 2d$.

We further extend ϕ to all of $\mathbb{R}$ via $\phi(s) = \infty$ if $s \geq 1 + 2d$. Finally define $V(q) \equiv \phi(U(q))$ for $q \in \mathbb{R}^n$ and

$$\bar{H}(z) \equiv K(z) + V(q) .$$

Thus $\bar{H}$ is C^2 where finite and if $\bar{H}(z) = 1$, $U(q) \leq V(q) \leq 1$ by (ϕ_2). Thus $V(q) = U(q)$ by (ϕ_1) and $\bar{H}(z) = H(z)$. Consequently $\bar{H}^{-1}(1) = \mathcal{D}$.

We will find a periodic solution of

$$\dot{p} = -\bar{H}_q, \quad \dot{q} = \bar{H}_p \tag{1.7}$$

on $\bar{H}^{-1}(1)$. Hence it will be a periodic solution of (HS) on $\mathcal{D}$.

To conclude this section, some estimates will be obtained for V. Let $\psi \in C^\infty(\mathbb{R},\mathbb{R})$ such that

$$\psi(s) = 0, \quad s < 1 - 2d$$
$$= 1, \quad s > 1 - d$$

and $\psi'(s) > 0$ if $s \in (1 - 2d, 1 - d)$. Set

$$\nu(q) = \psi(V(q)) \frac{U_q(q)}{|U_q(q)|} .$$

Observing that the ψ term vanishes if $q \in \Omega_{1-2d}$ and $\psi'(V(q))$ vanishes if $V(q) > 1 - d$, it follows that $\nu \in C^1(\mathbb{R}^n,\mathbb{R})$.

<u>Proposition 1.8</u>: There is a constant $\gamma > 0$ such that

$$V_q(q) \cdot \nu(q) \geq 0 \quad \text{for all} \quad q \in \Omega_{1+2d} \tag{1.9}$$

and

$$V_q(q) \cdot \nu(q) \geq \gamma V(q) \quad \text{for all} \quad q \in \Omega_{1+2d} \setminus \Omega_{1-2d} . \tag{1.10}$$

Proof: Inequality (1.9) is immediate from the definition of V and (ϕ_2). To check (1.10), note that if $U(q) \in (1 - d, 1 + 2d)$, then $V(q) > U(q)$ by (ϕ_2) and $\psi(V(q)) = 1$. Therefore

$$V_q(q) \cdot \nu(q) = \phi'(U(q))|U_q| > \beta(1 - d)\phi'(U(q)) \tag{1.11}$$

by (1.6). Thus to get (1.10), it suffices to show

$$\phi'(s) > \beta_1\phi(s), \quad s \in (1 - d, 1 + 2d) \tag{1.12}$$

and this is immediate from (ϕ_1)-(ϕ_3).

§2. Functional Analytical Preliminaries

The space in which (1.7) will be treated is the Hilbert space

$$E \equiv \{z = (p,q) \mid p \in L^2(S^1,\mathbb{R}^N),\ q \in W^{1,2}(S^1,\mathbb{R}^n)\}$$
$$\equiv L^2(S^1,\mathbb{R}^n) \oplus W^{1,2}(S^1,\mathbb{R}^n)$$

where $L^2(S^1,\mathbb{R}^n)$ denotes the set of n-tuples of 2π periodic functions which are square integrable, etc. For $w \in L^2(S^1,\mathbb{R}^n)$, let

$$[w] \equiv \frac{1}{2\pi}\int_0^{2\pi} w(t)dt \ .$$

Thus any $z = (p,q) \in E$ can be decomposed into $([p],[q]) + (\hat{p},\hat{q})$ where $(\hat{p},\hat{q}) \in \hat{L}^2 \oplus \hat{W}^{1,2}$ and

$$\hat{L}^2 \equiv \{p \in L^2(S^1,\mathbb{R}^n) \mid [p] = 0\} \ ,$$
$$\hat{W}^{1,2} \equiv \{q \in W^{1,2}(S^1,\mathbb{R}^n) \mid [q] = 0\} \ .$$

As inner product in E we take

$$(z_1,z_2)_E \equiv \int_0^{2\pi} [(\hat{p}_1(t) \cdot \hat{p}_2(t)) + (D\hat{q}_1 \cdot D\hat{q}_2)]dt + [p_1][p_2] + [q_1][q_2]$$

where $D \equiv \frac{d}{dt}$ and $z_1 = (p_1,q_1) = ([p_1] + \hat{p}_1, [q_1] + \hat{q}_1)$, etc. The norm in E will be denoted by $\|\cdot\|$ and we will generally use the same notation for the norm in E^*, the dual space of E.

It is easy to see that $D|_{\hat{W}^{1,2}} : \hat{W}^{1,2} \to \hat{L}^2$ is an isomorphism. Let D^{-1} denote its inverse. We define linear maps P^0, P^+, P^- of E into E by

$$P^0(p,q) \equiv ([p],[q])$$

and

$$P^{\pm}(p,q) \equiv \left(\frac{1}{2}(\hat{p} \pm D\hat{q}),\ \frac{1}{2}(\hat{q} \pm D^{-1}\hat{p})\right) \ .$$

It is easy to verify that these maps are well defined and are (continuous) projectors on

E satisfying $P^0 + P^+ + P^- = id$, the identity map on E. Define $E^0 \equiv P^0E$ and $E^\pm \equiv P^\pm E$. Note that if $(p^\pm, q^\pm) \in E^\pm$, then $p^\pm = \frac{1}{2}(p^\pm \pm Dq^\pm)$. Therefore

$$p^\pm = \pm Dq^\pm \ . \tag{2.1}$$

Next observe that the spaces E^0, $E^\pm$ are mutually orthogonal subspaces of E. E.g. if $z^\pm = (p^\pm, q^\pm) \in E^\pm$,

$$\begin{aligned}(z^+, z^-)_E &= \int_0^{2\pi} [(p^+ \cdot p^-) + (Dq^+ \cdot Dq^-)]dt \\ &= \int_0^{2\pi} [(Dq^+ \cdot (-Dq^-)) + (Dq^+ \cdot Dq^-)]dt = 0\end{aligned}$$

via (2.1).

For $z = (p,q) \in E$, define the action integral as

$$A(z) \equiv \int_0^{2\pi} p \cdot \dot{q} \, dt \ .$$

Then $A \in C^\infty(E,\mathbb{R})$ and writing $z = z^0 + z^+ + z^-$ and using (2.1) shows

$$\begin{aligned}A(z) &= \int_0^{2\pi} (p^0 + p^+ + p^-) \cdot (Dq^+ + Dq^-)dt \\ &= \int_0^{2\pi} [(p^+ \cdot Dq^+) + (p^+ \cdot Dq^-) + (p^- \cdot Dq^+) + (p^- \cdot Dq^-)]dt \\ &= \frac{1}{2}\int_0^{2\pi} (|p^+|^2 + |Dq^+|^2 - |p^-|^2 - |Dq^-|^2)dt \\ &= \frac{1}{2}(\|z^+\|^2 - \|z^-\|^2) \ .\end{aligned} \tag{2.2}$$

Next define

$$\Psi(z) \equiv \frac{1}{2\pi}\int_0^{2\pi} \bar{H}(z)dt$$

and

$$\mathcal{M} \equiv \{z \in E \mid \Psi(z) = 1\} .$$

Our goal is to obtain a periodic solution of (HS) (or equivalently (1.7)) as a critical point of $A|_{\mathcal{M}}$. As will be seen later, a critical point z of this constrained variational problem satisfies $z \in C^1(S^1, \mathbf{R}^{2n})$ and

$$\dot{z} = \lambda J \bar{H}_z(z) \tag{2.3}$$

where $\lambda \in \mathbf{R}\backslash\{0\}$. Since (2.3) is a Hamiltonian system, $\bar{H}(z(t)) \equiv$ constant. Thus $z \in \mathcal{M}$ implies $z \in \mathcal{D}$. Moreover since $\lambda \neq 0$, rescaling time in (2.3) yields a periodic solution of (1.7) on $\mathcal{D}$, i.e. the desired solution of (HS).

§3. Some Properties of $\mathcal{M}$ and $A|_{\mathcal{M}}$

This section studies several properties of Ψ and $\mathcal{M}$. It will be shown that $\mathcal{M}$ is a $C^{1,1}$ manifold which bounds a neighborhood of 0 in E and $A|_{\mathcal{M}}$ satisfies a version of the Palais-Smale condition.

For $q \in W^{1,2}(S^1,\mathbb{R}^n)$, set

$$\mathcal{V}(q) \equiv \frac{1}{2\pi}\int_0^{2\pi} V(q(t))dt .$$

Remark 3.1: The definition of V shows that if $\mathcal{V}(q) < \infty$, $q(t) \in \Omega_{1+2d}$ for almost all $t \in [0,2\pi]$ and since $q \in C(S^1,\mathbb{R}^n)$, $q(t) \in \bar{\Omega}_{1+2d}$ for all $t \in [0,2\pi]$. In particular there is a constant $M > 0$ (and independent of q) such that $\|q\|_{L^\infty} < M$.

For $x \in \mathbb{R}^n$, let

$$\ell(x) \equiv \inf_{y\in\partial\Omega_{1+2d}} |x - y| .$$

Proposition 3.2: There exist constants β_1, M_1 such that for all $x \in \bar{\Omega}_{1+2d}$,

$$\beta_1\ell(x)^{-2} \le V(x) + M_1 . \tag{3.3}$$

Proof: Let $x \in \Omega_{1+2d}$. Using the implicit function theorem, it is not difficult to show that there is an $\varepsilon_0 > 0$ such that if $\ell(x) < \varepsilon_0$, there exists a unique $\bar{x} \in \partial\Omega_{1+2d}$ and $\rho > 0$ such that

$$x = \bar{x} - \rho U_q(\bar{x}) .$$

Therefore there is a $\beta_2 > 0$ such that

$$\beta_2\rho \ge |x - \bar{x}| \ge \beta\rho \quad \text{if} \quad \ell(x) < \varepsilon_0 \tag{3.4}$$

via the continuity of U_q and (1.6). Now

$$U(x) = U(\bar{x}) = U_q(\bar{x})(x - \bar{x}) + o(|x - \bar{x}|) \tag{3.5}$$

as $x \to \partial\Omega_{1+2d}$. Therefore by (3.4) - (3.5), for x near $\partial\Omega_{1+2d}$, e.g. $\ell(x) \le \varepsilon$,

$$|U(x) - U(\bar{x})| \le \rho|U_q(\bar{x})|^2 + o(q) \le M_2\rho . \tag{3.6}$$

Now for $\ell(x) \le \varepsilon$, by (ϕ_3) and (3.6),

$$V(x) = (U(x) - (1 + 2d))^{-2} \ge (M_2\rho)^{-2} .$$

But $\rho = \ell(x)|U_q(\bar{x})|^{-1}$. Therefore

$$(3.7) \qquad V(x) \geq M_3 \ell(x)^{-2}$$

if $\ell(x) \leq \varepsilon$. If $\ell(x) > \varepsilon$, $\ell(x)^{-2} \leq \varepsilon^{-2}$ so

$$V(x) + M_3\varepsilon^{-2} \geq M_3 \ell(x)^{-2} .$$

Thus (3.3) obtains with $M_2 = M_3\varepsilon^{-2}$ and $\beta_1 = M_3$.

The estimate (3.3) will be used next to show that $V(q) < \infty$ implies that $q(t)$ avoids $\partial\Omega_{1+2d}$.

Proposition 3.8: Let $q \in W^{1,2}(S^1,\mathbb{R}^n)$ satisfy $V(q) < \infty$. Then there is an $\bar{M} = \bar{M}(\|q\|_{W^{1,2}}, V(q)) > 0$ such that $\ell(q(t)) \geq \bar{M}$ for all $t \in [0,2\pi]$.

Proof: Since q is 2π periodic, by translating t it can be assumed that

$$\ell(q(0)) = \min_{t\in[0,2\pi]} \ell(q(t)) \equiv \mu .$$

By the Cauchy-Schwarz inequality,

$$(3.9) \qquad |q(t) - q(0)| \leq \int_0^t |\dot{q}(\tau)|d\tau \leq t^{1/2}\|q\|_{W^{1,2}} .$$

Since ℓ is Lipschitz continuous (with constant 1),

$$(3.10) \qquad |\ell(q(t)) - \ell(q(0))| \leq |q(t) - q(0)| \leq t^{1/2}\|q\|_{W^{1,2}} .$$

Therefore

$$(3.11) \qquad \ell(q(t)) \leq \mu + t^{1/2}\|q\|_{W^{1,2}} .$$

We can assume $\|q\|_{W^{1,2}} > 0$ for otherwise the result is trivial. By (3.3) and (3.11),

$$(3.12) \qquad \infty > \mathcal{V}(q) \geq \frac{1}{2\pi}\int_0^{2\pi} (\beta_1 \ell(q(t))^{-2} - M_1)dt$$

$$\geq \frac{\beta_1}{2\pi}\int_0^{2\pi} (\mu + t^{1/2}\|q\|_{W^{1,2}})^{-2}dt - M_1$$

$$\geq \frac{\beta_1}{\pi}\int_0^{2\pi} (\mu^2 + t\|q\|^2_{W^{1,2}})^{-1}dt - M_1$$

$$= \frac{\beta_1}{\pi\|q\|^2_{W^{1,2}}} \log\left(1 + \frac{2\pi\|q\|^2_{W^{1,2}}}{\mu^2}\right) - M_1$$

from which the result follows.

Remark 3.13: Proposition 3.8 implies that the domain of $\mathcal{V}$ is $\{q \in W^{1,2}(S^1,\mathbb{R}^n) \mid q(t) \in \Omega_{1+2d}$ for all $t \in [0,2\pi]\}$.

The smoothness of $\mathcal{V}$ will be established next.

Proposition 3.14: $\mathcal{V} \in C^2$ on the domain of $\mathcal{V}$.

Proof: Let $q \in W^{1,2}(S^1,\mathbb{R}^n)$ with $\mathcal{V}(q) < \infty$. Let $\delta = \inf_{t\in[0,2\pi]} \ell(q(t))$. Then $\delta > 0$ by Proposition 3.8. Let $\tilde{q} \in W^{1,2}(S^1,\mathbb{R}^n)$ with $\|\tilde{q}\|_{W^{1,2}} < \rho$. If ρ is sufficiently small, $\|\tilde{q}\|_{L^\infty} < \delta/2$. Therefore since $V \in C^2(\Omega_{1+2d},\mathbb{R})$,

$$(3.15) \qquad V(q + \tilde{q}) = V(q) + V_q(q)\tilde{q} + \frac{1}{2} V_{qq}(q)(\tilde{q},\tilde{q}) + o(|\tilde{q}|^2)$$

as $\tilde{q} \to 0$ uniformly for $t \in [0,2\pi]$. The definition of $\mathcal{V}$, (3.15), and the compact embedding of $W^{1,2}(S^1,\mathbb{R}^n)$ in $C(S^1,\mathbb{R}^n)$ then readily imply that $\mathcal{V}$ is Frechet differentiable at q with

$$\mathcal{V}'(q)\tilde{q} = \frac{1}{2\pi}\int_0^{2\pi} V'(q)\tilde{q}\, dt ,$$

$\mathcal{V}'(q)$ is continuous, $(\mathcal{V}'(q))' \equiv \mathcal{V}''(q)$ exists,

$$\mathcal{V}''(q)(\tilde{q},\tilde{q}) = \frac{1}{2\pi}\int_0^{2\pi} V''(q)(\tilde{q},\tilde{q})dt$$

and $\mathcal{V}''(q)$ is continuous.

For $z \in E$, set

$$\mathcal{K}(z) \equiv \frac{1}{2\pi}\int_0^{2\pi} K(z)dt .$$

Proposition 3.16: $\mathcal{K} \in C^{1,1}(E,\mathbb{R})$, (i.e. $\mathcal{K}$ is Frechet differentiable and its Frechet derivative is Lipschitz continuous).

Proof: Since $K \in C^2(\mathbb{R}^{2n},\mathbb{R})$, given any $z,\zeta \in \mathbb{R}^{2n}$, by Taylor's Theorem,

$$K(z + \zeta) = K(z) + K_z(z)\zeta \frac{1}{2} K_{zz}(z + \theta\zeta)(\zeta,\zeta) \tag{3.17}$$

for some $\theta \in (0,1)$. By (K_6) of Proposition 1.3, K_{zz} is uniformly bounded. Therefore there is a constant $M_3 > 0$ such that

$$|K(z + \zeta) - K(z) - K_z(z)\zeta| \le M_3|\zeta|^2 \tag{3.18}$$

for all $z,\zeta \in \mathbb{R}^{2n}$. Choosing $z,\zeta \in E$, (3.18) implies

$$|\mathcal{K}(z + \zeta) - \mathcal{K}(z) - \frac{1}{2\pi}\int_0^{2\pi} K_z(z)\zeta\, dt| \le \frac{M_3}{2\pi} \|\zeta\|^2_{L^2} \le M_4\|\zeta\|^2 . \tag{3.19}$$

In particular for z fixed, given any $\varepsilon > 0$, if ζ is sufficiently small, the right hand side of (3.19) does not exceed $\varepsilon\|\zeta\|$. Hence $\mathcal{K}$ is Frechet differentiable and

$$\mathcal{K}'(z)\zeta = \frac{1}{2\pi}\int_0^{2\pi} K_z(z)\zeta dt .$$

To show that $\mathcal{K}'$ is Lipschitz continuous, note that

$$\|\mathcal{K}'(z + w) - \mathcal{K}'(z)\|_{E^*} = \sup_{\zeta\in E,\|\zeta\|\le 1} |\frac{1}{2\pi}\int_0^{2\pi} (K_z(z + w) - K_z(z))\zeta dt . \tag{3.20}$$

As in (3.17) by the Mean Value Theorem and (K_6),

$$|K_z(z + w) - K_z(z)| \le M_5|w| \tag{3.21}$$

for some constant M_5. Therefore (3.20)-(3.21) imply

(3.22) $$\|K'(z + w) - K'(z)\|_{E^*} \le M_6\|w\| ,$$

i.e. K' is Lipschitz continuous.

By the definitions of K and V, $\Psi = K + V$. Recall that $M = \Psi^{-1}(1)$. The next three propositions study some properties of M.

<u>Proposition 3.23:</u> M is $C^{1,1}$ manifold in E.

<u>Proof:</u> The smoothness assertions follow on combining Proposition 3.14 and 3.16 once we show that M is a manifold, i.e. $\Psi'(z) \neq 0$ for all $z \in M$. But if $z = (p,q)$ with $p \not\equiv 0$,

$$\Psi'(z)(p,0) = \frac{1}{2\pi}\int_0^{2\pi} p \cdot K_p(z)dt > 0$$

via (K_2) of Proposition 1.3. If $p \equiv 0$, then $\Psi(z) = V(q) = 1$ via (K_1) and by Proposition 3.8, $q(t) \in \Omega_{1+2d}$ for all $t \in [0,2\pi]$. If $q(t) \in \Omega_{1-d}$ for all $t \in [0,2\pi]$, $V(q) \le 1 - d$ which is impossible. Thus since $q \in C(S^1,\mathbb{R}^n)$, $q(t) \in \Omega_{1+2d}\backslash\Omega_{1-d}$ on a set Y of positive measure. By previous remarks, $\nu(q) \in C(S^1,\mathbb{R}^n)$ and $(0,\nu(q)) \in E$. Hence by (1.9)-(1.10),

(3.24) $$\Psi'(z)(0,\nu(q)) = \ '(q)\nu(q) \ge \frac{1}{2\pi}\int_Y \gamma(1 - d)dt > 0 .$$

Thus M is a manifold and the proof is complete.

<u>Proposition 3.25:</u> M is the boundary of a neighborhood of 0 in E.

<u>Proof:</u> Ψ is continuous on $L^2(S^1,\mathbb{R}^n) \oplus \{q \in W^{1,2}(S^1,\mathbb{R}^n)|V(q) < \infty\}$ which is an open set in E. Therefore $\Psi^{-1}(-\infty,1)$ is open. Since $\bar{H}(0) = 0$, 0 belongs to this set.

<u>Proposition 3.26:</u> M is bounded in $L^2(S^1,\mathbb{R}^{2n})$.

<u>Proof:</u> Let $(p,q) \in M$. Since $V(q) < \infty$, by Remark 3.1, there is an $M > 0$ such that $\|q\|_{L^\infty} \le M$. The definition of K implies it is a multiple of $|p|^2$ for $|z|$ near ∞. Hence there are constants $M_7,M_8 > 0$ such that

$$K(z) \ge M_7|p|^2 - M_8$$

for all $z \in \mathbb{R}^{2n}$. Therefore

$$K(z) \geq \frac{M_7}{2\pi} \|p\|^2_{L^2} - \frac{M_8}{2\pi} .$$

Thus if $z \in M$,

$$\|p\|^2_{L^2} \leq \left(1 + \frac{M_8}{2\pi}\right) \frac{2\pi}{M_7} .$$

The next proposition shows that $A|_M$ satisfies a version of the Palais-Smale condition.

Proposition 3.27: $A|_M$ satisfies $(PS)^+$, i.e. if $c > 0$ and (z_j) is a sequence in E such that

(i) $z_j \in M$,

(ii) $A(z_j) \to c$,

and (iii) $A'(z_j) - \lambda_j \Psi'(z_j) \to 0$ (in E^*)

as $j \to \infty$ where

$$\lambda_j = (A'(z_j),\Psi'(z_j))_{E^*} \|\Psi'(z_j)\|^{-2}$$

then (z_j) has a convergent subsequence.

Proof: By (ii), there is an $\varepsilon \in (0, \frac{c}{2})$ such that

(3.28) $$c - \varepsilon \leq A(z_j) \leq c + \varepsilon$$

for all large $j \in \mathbb{N}$. Similarly by (iii), there is a $w_j \in E^*$ with $w_j \to 0$ as $j \to \infty$ and

(3.29) $$A'(z_j)\zeta - \lambda_j \Psi'(z_j)\zeta = \langle w_j,\zeta\rangle$$

where $\langle \cdot,\cdot \rangle$ denotes the duality between E^* and E. Choosing $\zeta = (p_j,0)$ yields

(3.30) $$A(z_j) - \frac{\lambda_j}{2\pi} \int_0^{2\pi} K_p(z_j)p_j dt = \langle w_j,(p_j,0)\rangle .$$

By (3.28) and the choice of ε, $A(z_j) \geq \frac{c}{2}$ while the left hand side of (3.30) goes to 0

as $j \to \infty$ since $((p_j,0))$ is bounded in E. Consequently by (K_2)-(K_3) of Proposition 1.3, λ_j is positive and bounded away from 0 for large j. We restrict ourselves to such $j \in \mathbb{N}$.

Choosing $\zeta = (\xi,\eta)$, (3.29) can be rewritten as

$$(3.31) \quad \int_0^{2\pi} [(p_j \cdot \dot{\eta}) + (\xi \cdot \dot{q}_j)]dt - \frac{\lambda_j}{2\pi}\int_0^{2\pi} [(\bar{H}_p(z_j) \cdot \xi) + (\bar{H}_q(z_j) \cdot \eta)]dt = \langle w_j,\zeta\rangle .$$

Setting $w_j = (u_j,v_j) \in E^*$, i.e. $u_j \in L^2(S^1,\mathbb{R}^n)$ and $v_j \in W^{-1,2}(S^1,\mathbb{R}^n)$ and noting that $\dot{p}_j \in W^{-1,2}(S^1,\mathbb{R}^n)$, (3.31) implies that

$$(3.32) \qquad \text{(i)} \quad -\dot{p}_j = \frac{\lambda_j}{2\pi}\bar{H}_q(z_j) + v_j$$

$$\text{(ii)} \quad \dot{q}_j = \frac{\lambda_j}{2\pi}\bar{H}_p(z_j) + u_j = \frac{\lambda_j}{2\pi}K_p(z_j) + u_j$$

holds in the sense of distributions.

We claim that

$$(3.33) \qquad \int_0^{2\pi} K_p(z_j) \cdot p_j dt \geq a > 0$$

for all $j \in \mathbb{N}$. Assuming (3.33) for now, (3.30) and (3.28) then imply λ_j is bounded away from ∞. Then by (3.32) (ii),

$$(3.34) \qquad \|\dot{q}_j\|_{L^2} \leq \frac{\lambda_j}{2\pi}\|K_p(z_j)\|_{L^2} + \|u_j\|_{L^2}$$

so (K_3) of Proposition 1.3, the boundedness of (z_j) in L^2 (via Proposition 3.26) and Remark 3.1 show that (q_j) is bounded in $W^{1,2}(S^1,\mathbb{R}^n)$. Proposition 3.8 then implies there is an $\bar{M} > 0$ independent of j such that $\ell(q_j(t)) \geq \bar{M}$ for all $t \in [0,2\pi]$, i.e. the functions q_j lie uniformly inside Ω_{1+2d}. Therefore the functions $(V_q(q_j))$ are bounded in L^∞ and by (K_5) of Proposition 1.3, $\bar{H}_q(z_j)$ are bounded in L^2. Consequently

the right hand side of (3.32)(i) converges strongly in $W^{-1,2}(S^1,\mathbb{R}^n)$ (along a subsequence). Therefore $\dot{p}_j$ converges strongly in $W^{-1,2}(S^1,\mathbb{R}^n)$. Consequently $D^{-1}\dot{p}_j$ converges strongly in $L^2(S^1,\mathbb{R}^n)$. But $p_j = [p_j] + D^{-1}\dot{p}_j$. Hence along a subsequence p_j converges strongly in $L^2(S^1,\mathbb{R}^n)$. Lastly by (3.32) (ii), the same is true for a subsequence of q_j in $W^{1,2}(S^1,\mathbb{R}^n)$ since (K_3) of Proposition 1.3 implies that K_p is a continuous map of $L^2(S^1,\mathbb{R}^{2n})$ to $L^2(S^1,\mathbb{R}^n)$.

Thus Proposition 3.27 will be established once we show that (3.33) holds. Suppose that (3.33) is false, i.e.

$$(3.35) \qquad \int_0^{2\pi} K_p(z_j) \cdot p_j dt \to 0$$

for some subsequence of j's. This implies that $\|p_j\|_{L^2} \to 0$ along this subsequence. Indeed let $Y_{1j} = \{t \in [0,2\pi] \mid |p_j(t)| < \sigma\}$, $Y_{2j} = \{t \in [0,2\pi] \mid \sigma \le |p_j(t)| \le 4\beta\}$, and $Y_{3j} = \{t \in [0,2\pi] \mid |p_j(t)| > 4\beta\}$ where $\sigma < 1$ and β was defined together with $\hat{H}$ in §1. On Y_{3j}, $p_j \cdot K_p(z_j) = 2\tilde{\rho}|p_j|^2$. On Y_{2j}, both $|p|^2$ and $p \cdot K_p$ are bounded away from 0. Therefore there is a constant $\alpha = \alpha(\sigma)$ such that $|p|^2 \le \alpha(\sigma)p \cdot K_p$ on Y_{2j}. Combining these observations yields

$$(3.36) \qquad \int_0^{2\pi} |p_j|^2 dt \le 2\pi\sigma^2 + \alpha(\sigma) \int_{Y_{2j}} p_j \cdot K_p(z_j)dt + (2\hat{\rho})^{-1} \int_{Y_{3j}} p_j \cdot K_p(z_j)dt .$$

Since σ is arbitrary, (3.35)-(3.36) show $\|p_j\|_{L^2} \to 0$ as $j \to \infty$. Then (K_3)-(K_4) of Proposition 1.3 imply

$$(3.37) \qquad \mathcal{K}(z_j) \to 0$$

as $j \to \infty$ while (K_3) and Proposition 3.26 show

$$(3.38) \qquad \int_0^{2\pi} |K_p(z_j)||p_j| dt \to 0$$

as $j \to \infty$. Also by (K_5),

$$(3.39) \qquad \left|\int_0^{2\pi} K_q(z_j) \cdot \nu(q_j)dt\right| \le \|K_q(z_j)\|_{L^2}\|\nu(q_j)\|_{L^2} \to 0$$

as $j \to \infty$ since $(\|\nu(q_j)\|_{L^2})$ is uniformly bounded. By (3.37) and (i) of Proposition 3.27,

$$(3.40) \qquad V(q_j) \to 1$$

as $j \to \infty$. Set

$$\Delta_j \equiv \{t \in [0,2\pi] \,|\, V(q_j(t)) > 1 - d\} .$$

Therefore

$$(3.41) \qquad V(q_j) \le \frac{1}{2\pi} \left(\int_{\Delta_j} V(q_j(t))dt + \int_{[0,2\pi]\setminus \Delta_j} V(q_j(t))dt\right)$$

$$\le \frac{1}{2\pi} \int_{\Delta_j} V(q_j(t))dt + 1 - d .$$

Hence for large j, by (3.40)-(3.41),

$$(3.42) \qquad \frac{d}{2} \le \frac{1}{2\pi} \int_{\Delta_j} V(q_j(t))dt .$$

Next by (3.32) (i) and (1.9),

$$(3.43) \qquad \int_0^{2\pi} p_j \cdot \frac{d}{dt} \nu(q_j(t))dt = \frac{\lambda_j}{2\pi} \int_0^{2\pi} \bar{H}_q(z_j) \cdot \nu(q_j)dt + \langle v_j,\nu(q_j)\rangle$$

$$\ge \frac{\lambda_j}{2\pi} \left[\int_{\Delta_j} V_q(q_j) \cdot \nu(q_j)dt - \|K_q(z_j)\|_{L^2}\|\nu(q_j)\|_{L^2}\right] - \|v_j\|_{W^{-1,2}}\|\nu(q_j)\|_{W^{1,2}} .$$

As has been noted earlier, $\nu(\cdot) \in C^1(\mathbb{R}^n,\mathbb{R})$ and $\|q\|_{L^\infty} \le M$ for all $(p,q) \in M$. Hence there exists a constant $M_1 > 0$ and independent of j such that

$$\|\nu(q_j)\|_{W^{1,2}} \le M_1(1 + \|\dot{q}_j\|_{L^2}) .$$

Thus (3.42)-(3.43), (1.10), and (K_5) of Proposition 1.3 imply that

$$(3.44) \qquad \int_0^{2\pi} p_j \cdot \frac{d}{dt} \nu(q_j)dt \geq \lambda_j \gamma \frac{d}{2} - o(1) - o(1)\|\dot{q}_j\|_{L^2}$$

as $j \to \infty$. On the other hand, by (3.32) and (3.38),

$$(3.45) \qquad \int_0^{2\pi} p_j \cdot \frac{d}{dt} \nu(q_j)dt \leq \|\nu'(q_j)\|_{L^\infty} \int_0^{2\pi} |p_j||\dot{q}_j|dt$$

$$\leq M_2(\lambda_j \int_0^{2\pi} |K_p(z_j)||p_j|dt + \int_0^{2\pi} |u_j||p_j|dt$$

$$\leq \lambda_j o(1) + o(1)^2$$

as $j \to \infty$. Combining (3.44)-(3.45) shows

$$(3.46) \qquad \frac{\gamma d}{4} \lambda_j \leq o(1)\|\dot{q}_j\|_{L^2} + o(1)$$

as $j \to \infty$. But then by (3.32) (ii),

$$(3.47) \qquad \frac{\gamma d}{4} \lambda_j \leq o(1)\lambda_j \|K_p(z_j)\|_{L^2} + o(1)$$

so (K_3) of Proposition 1.3 and Proposition 3.26 imply $\lambda_j \to 0$ as $j \to \infty$, a contradiction. Thus (3.33) has been verified and Proposition 3.27 has been established.

Two further technical results are needed in this section. Let L denote the duality map between $W^{-1,2}(S^1,\mathbb{R}^n)$ and $W^{1,2}(S^1,\mathbb{R}^n)$, i.e. L is defined by

$$(Lw,\xi)_{W^{1,2}} = \langle w,\xi \rangle$$

for $w \in W^{-1,2}(S^1,\mathbb{R}^n)$ and $\xi \in W^{1,2}(S^1,\mathbb{R}^n)$. Abusing notation somewhat, we will also let L denote the duality between E and E^*. For $z = (p,q) \in E$, define $P_1 z \equiv p$ and $P_2 z \equiv q$.

<u>Proposition 3.48</u>: $P_2 L \Psi'$ is a compact map of M into $W^{1,2}(S^1,\mathbb{R}^n)$.

Proof: For $z \in M$ and $\zeta = (u,v) \in E$,

$$\Psi'(z)\zeta = \langle \Psi'(z),\zeta \rangle = \frac{1}{2\pi}\int_0^{2\pi} [(\bar{H}_p(z) \cdot u) + (\bar{H}_q(z) \cdot v)]dt = (\zeta\Psi'(z),\zeta)_E .$$

Therefore

$$(3.49) \qquad (P_2 L\Psi'(z),v)_{W^{1,2}} = \frac{1}{2\pi}\int_0^{2\pi} \bar{H}_q(z) \cdot v dt .$$

The right hand side of (3.49) is a continuous linear functional on $W^{1,2}(S^1,\mathbf{R}^n)$. Therefore there exists a unique $\Theta = \Theta(z) \in W^{1,2}(S^1,\mathbf{R}^n)$ such that

$$(3.50) \qquad (\Theta(z),v)_{W^{1,2}} = \frac{1}{2\pi}\int_0^{2\pi} \bar{H}_q(z) \cdot v dt .$$

Clearly $\Theta(z) = P_2 L\Psi'(z)$. But the map $z \to \bar{H}_q(z)$, $M \to L^2(S^1,\mathbf{R}^n)$ is continuous and the map $\bar{H}_q(z) \to \Theta(z)$, $L^2(S^1,\mathbf{R}^n) \to W^{1,2}(S^1,\mathbf{R}^n)$ is compact and linear. It follows that $P_2 L\Psi'$ is compact.

The final result in this section is *a version of the so-called "Deformation Theorem"* which is appropriate for our setting. A subset $S \subset E$ will be called invariant if $z(t) \in S$ implies that $z(t+\theta) \in S$ for all $t \in [0,2\pi]$. A mapping $h : S \to E$, where S is invariant, will be called equivariant if $h(T_\theta z) = T_\theta h(z)$ for all $\theta \in [0,2\pi]$ where $T_\theta\zeta(t) = \zeta(t+\theta)$. For $s \in \mathbf{R}$, let

$$A_s \equiv \{z \in M \mid A(z) > s\} .$$

For $c \in \mathbf{R}$, let

$$K_c \equiv \{z \in M \mid A(z) = c \text{ and } A'(z) = (A'(z),\Psi'(z))_{E^*}\|\Psi'(z)\|^{-2}\Psi'(z)\}$$

i.e. K_c is the set of critical points of $A|_M$ having critical value c.

Proposition 3.51: Let $c, \bar{\varepsilon} > 0$. Then there is an $\varepsilon \in (0, \bar{\varepsilon})$ and $\eta \in C([0,1] \times M, M)$ such that

1° $\eta(1,\cdot)$ is equivariant

2° $\eta(1,z) = z$ if $A(z) \notin [c - \bar{\varepsilon}, c + \bar{\varepsilon}]$

3° $\|\eta(1,z) - z\| \leq 1$

4° If $K_c = \emptyset$, $\eta(1, A_{c-\varepsilon}) \subset A_{c+\varepsilon}$

5° $P^+\eta(1,z) = \beta^+(z)z^+ + B^+(z)$ where $\beta^+ \in C(M,[1,e])$ and P_2B^+ is compact.

Proof: Most of the above assertions follow from standard arguments and therefore we will be somewhat sketchy below. See e.g. [13-15] for more details. The function η is determined as the solution of an ordinary differential equation of the form

$$\frac{d\eta}{dt} = \omega(\eta) L[A'(\eta) - \frac{(A'(\eta), L'(\eta))_E^*}{\|\Psi'(\eta)\|^2} \Psi'(\eta)] \tag{3.52}$$

$$\eta(0,z) = z \in E .$$

The scalar function ω is Lipschitz continuous, $0 \leq \omega(z) \leq 1$, $\omega(z) = 1$ if $z \in M$ and $A(z)$ is near c. Note that the argument of Proposition 3.23 shows $\Psi^{-1}(s)$ is a manifold for each s near 1, e.g. $|s - 1| \leq s_0$. The function $\omega(z) = 0$ if $|\Psi(z) - 1| > s_0$. Lastly $\omega(T_\theta z) = \omega(z)$ for all $\theta \in [0,2\pi]$.

Since the right hand side of (3.52) is Lipschitz continuous and is bounded by 1 (see [14] or [15]), there exists a solution of (3.52) defined for all $t \in \mathbf{R}$ and $z \in E$. Moreover $\|\eta(t,z) - z\| \leq 1$ for $t \in [0,1]$, i.e. 3° holds. The form of (3.52) implies that $\eta(t,M) = M$ for all $t \in \mathbf{R}$. The properties of ω show that $\eta(1,\cdot)$ satisfies 1°-2°. Proposition 3.27 and a standard argument - see [13]-[15] imply 4°. To prove 5°, note that $P^+LA'(z) = z^+$. Therefore integrating (3.52) yields:

$$(3.53) \qquad P^+\eta(t,z) = (\exp \int_0^t \omega(\eta(s,z))ds)z^+ -$$

$$- \int_0^t (\exp \int_0^\tau \omega(\eta(s,z))ds)\omega(\eta(\tau,z))(A'(\eta(\tau,z),\Psi'(\eta(\tau,z)))_{E^*}$$

$$\|\Psi'(\eta(\tau,z))\|^{-2}P^+L\Psi'(\eta(\tau,z))d\tau \ .$$

Thus $P^+\eta$ has the form stated in 5^o. The compactness of P_2B^+ follows via Proposition 3.48 and an argument from [16] since P_2 and P^+ commute.

Remark 3.54: Let $A_s = \{z \in M | A(z) \leq s\}$. If we replace $\omega(z)$ by $-\omega(z)$ in (3.52), the assertions of Proposition 3.51 still hold with 4^o replace by $\eta(1,A_{c+\varepsilon}) \subset A_{c-\varepsilon}$ and 5^o by $P^-\eta(1,z) = \beta^-(z)z^- + B^-(z)$ where $\beta^- \in C(M,[e^{-1},1])$ and P_2B^- is compact.

§4. Existence of a Solution

The proof of Theorem 1 will be completed in this section. The solution will be obtained as a critical point of $A|_M$ by a minimax argument. Then a simple regularity argument shows it is a classical solution of (HS). The following two lemmas pave the way for the definition of the critical value c.

Lemma 4.1: Let $M^+ \equiv M \cap E^+$ and set

$$\underline{\alpha} \equiv \inf_{z \in M^+} A(z) .$$

Then $\underline{\alpha} > 0$.

Proof: By (2.2) for $z = z^+ \in E^+$, $A(z) = \frac{1}{2}\|z^+\|^2$. Since by Proposition 3.25 M is the boundary of a neighborhood of 0 in E, there is an $r > 0$ such that $\|z\| < r$ implies z is interior to M. In particular for $z \in \partial B_r(0) \cap E^+$, $A(z) > \frac{1}{2} r^2$. Hence $\underline{\alpha} > \frac{1}{2} r^2$.

Next let L^+ be a two dimensional invariant subspace of E^+. We further require that L^+ be such that there is a constant $a_1 > 0$ satisfying

$$\|z^+\|_{L^2} \le a_1 \|z\|_{L^2} \tag{4.2}$$

for all $z \in E^0 \oplus E^- \oplus L^+$. To find such an L^+, let $e_1,\ldots,e_n$ denote the usual basis in $\mathbb{R}^n$. Then we can take

$E^0 \equiv \mathrm{span}\{(e_j,0),(0,e_k) \mid 1 \le j,k \le n\}$,

$E^+ \equiv \mathrm{span}\{((j+1)\sin jt\, e_k - (1+\frac{1}{j})\cos jt\, e_k),$

$(k+1)\cos \ell t\, e_m,(1+\frac{1}{j})\sin \ell t\, e_m) \mid 1 \le k,m \le n,$ and $j,\ell \in \mathbb{N}\}$,

$E^- \equiv \mathrm{span}\{((j+1)\sin jt\, e_k,(1+\frac{1}{j})\cos jt\, e_k),$

$((\ell+1)\cos \ell t\, e_m, -(1+\frac{1}{\ell})\sin \ell t\, e_m) \mid 1 \le k,m \le n$ and $j,\ell \subset N\}$,

and $L^+ = \mathrm{span}\{(\sin t\, e_1,-\cos t\, e_1),(\cos t\, e_1,\sin t\, e_1)\}$.

It is easy to verify that (4.2) holds.

Lemma 4.3: If $M^- \equiv M \cap (E^- \oplus E^0 \oplus L^+)$ and

$$\bar{\alpha} \equiv \sup_{z \in M^-} A(z) ,$$

then $\bar{\alpha} < \infty$.

Proof: By Proposition 3.26, M is bounded in $L^2(S^1, R^{2n})$. Therefore there is an $M_1 > 0$ such that $\|z\|_{L^2} \leq M_1$ for all $z \in M$. In particular for $z = z^- + z^0 + z^+ \in M^-$, by (4.2) we have

$$\|z^+\|_{L^2} \leq a_1 M_1 . \tag{4.4}$$

Since L^+ is finite dimensional, there is a constant $a_2 > 0$ such that $\|z^+\| \leq a_2 \|z^+\|_{L^2}$ for all $z^+ \in L^+$. Hence

$$A(z) = \frac{1}{2} (\|z^+\|^2 - \|z^-\|^2) \leq \frac{1}{2} (a_1 a_2 M_1)^2 \tag{4.5}$$

for $z \in M^-$ and $\bar{\alpha} \leq \frac{1}{2} (a_1 a_2 M_1)^2$.

Now the class of sets that will be used to find a critical point of $A|_M$ can be introduced. Let

$\Gamma \equiv \{h \in C(M, M) \mid 1^0$ h is equivariant,

2^0 $h(z) = z$ if $A(z) \notin [0, \bar{\alpha} + 1]$,

3^0 $h(z)$ maps bounded sets to bounded sets

4^0 $P^+ h(z) = \beta(z) z^+ + B(z)$ where $\beta \in C(M, [1, \beta_0])$, $\beta_0 = \beta_0(h) > 1$, and $P_2 B(z)$ is compact$\}$.

A critical value c of $A|_M$ can be produced by taking:

$$c \equiv \sup_{h \in \Gamma} \inf_{z \in M^+} A(h(z)) . \tag{4.6}$$

To see this, note first that $id \in \Gamma$. Hence by Lemma 4.1, $c \geq \underline{\alpha} > 0$. To prove that $c < \infty$, the following intersection theorem which is of independent interest is required.

Theorem 4.7: Let $h \in \Gamma$. Then $h(M^+) \cap M^- \neq \emptyset$.

Proof: We will use a finite dimensional approximation argument. Let E_m^+ and E_m^- be $2m$ dimensional invariant subspaces of E^+, E^- respectively such that if $E_m = E_m^- \oplus E^0 \oplus E_m^+$, $\overline{\bigcup_{m\in\mathbb{N}} E_m} = E$. Such subspaces can be written down explicitly using the basis for E given following (4.2). Let P_m denote the orthogonal projector of E onto E_m. Set $h_m \equiv P_m h \in C(M^+ \cap E_m, E_m)$. Note that by properties 1°-2° of Γ, h_m is equivariant and $h_m(z) = z$ on $E^0 \cap M$. By Proposition 2.2 of [17] (where we take f to be the orthogonal projector of E^+ onto the orthogonal complement of L^+ in E^+ composed with h_m), there is a point $z_m \in M^+ \cap E_m$ such that $h_m(z_m) \in E^- \oplus E^0 \oplus L^+$. We claim (z_m) is a bounded sequence. Otherwise $\|z_m\| \to \infty$ along a subsequence. But then since $z_m \in E^+$, $A(z_m) = \frac{1}{2}\|z_m\|^2 \to \infty$. Hence by property 2° of Γ, $h_m(z_m) = z_m$ for large m. Therefore $z_m \in M^+ \cap (E^- \oplus E^0 \oplus L^+) = M \cap L^+$. Since M is bounded in L^2 and L^+ is finite dimensional, (z_m) must be bounded in E, a contradiction.

Thus (z_m) is a bounded sequence. By property 3° of Γ, $(h_m(z_m))$ is also bounded. Property 4° of Γ implies that

$$q_m = \beta(z_m)^{-1}(P_2 P^+ h_m(z_m) - P_2 B(z_m)) \tag{4.8}$$

Where $z_m = (p_m, q_m)$. The boundedness of (z_m) and compactness of $P_2 B$ show the second term on the right hand side of (4.8) has a convergent subsequence. The boundedness of $h_m(z_m)$ and the fact that $P^+ h_m(z_m)$ lies in L^+ which is finite dimensional implies the first term on the right hand side of (4.8) also has a convergent subsequence. It follows then from (4.8) that q_m has a convergent subsequence in $W^{1,2}(S^1, \mathbb{R}^n)$. Therefore the same is true for $p_m = Dq_m$ in $L^2(S^1, \mathbb{R}^n)$. Consequently $z_m \to z \in M^+$ and by the continuity of h, $h_m(z_m) \to h(z) \in M^-$. The Theorem is proved.

Corollary 4.9: $c \le \overline{\alpha} < \infty$.

Proof: By Theorem 4.7, $h(M^+) \cap M^- \ne \emptyset$ for any $h \in \Gamma$. Therefore for each $h \in \Gamma$,

$$\inf_{z\in M^+} A(h(z)) \le \sup_{w\in M^-} A(w) = \overline{\alpha}$$

via Lemma 4.3.

Now we can prove

Theorem 4.10: c is a critical value of $A|_M$.

Proof: If not, we can invoke Proposition 3.51 with $\bar{\varepsilon} = \frac{1}{2}\min(1,\underline{\alpha})$ obtaining $\eta(1,\cdot) \in C(M,M)$ and satisfying 1°-5° of Proposition 3.51. But 1°-3°, 5°, and our choice of $\bar{\varepsilon}$ imply that $\eta(1,\cdot) \in \Gamma$ as is $\eta(1,h)$ for any $h \in \Gamma$. By 4° of Proposition 3.51,

$$\eta(1,\cdot) : A_{c-\varepsilon} \to A_{c+\varepsilon} . \tag{4.11}$$

Choose $h \in \Gamma$ so that

$$\inf_{z\in M^+} A(h(z)) \geq c - \varepsilon . \tag{4.12}$$

By (4.11),

$$\inf_{z\in M^+} A(\eta(1,h(z))) \geq c + \varepsilon . \tag{4.13}$$

But since $\eta(1,h) \in \Gamma$, (4.6) shows

$$\inf_{z\in M^+} A(\eta(1,h(z)) \leq c , \tag{4.14}$$

a contradiction. Thus c is a critical value of $A|_M$.

Now finally we can complete the

Proof of Theorem 1: Since c is a critical value of $A|_M$, there is a $\lambda \in \mathbb{R}$ and $z \in M$ such that $A(z) = c$ and $A'(z) - \lambda\Psi'(z) = 0$, i.e.

$$\int_0^{2\pi} [(p \cdot \dot{Q}) + (P \cdot \dot{q}) - \frac{\lambda}{2\pi} (\bar{H}_p(z) \cdot P) + (\bar{H}_q(z) \cdot Q)]dt = 0 \tag{4.15}$$

for all $(P,Q) \in E$. Equation (4.15) expresses the fact that z is a weak solution of (2.3). The argument of (3.28)-(3.30) and our lower bound for c show $\lambda > 0$. A simple regularity argument - see the proof of Theorem 3.3 of [18] - shows $z \in C^1(S^1,\mathbb{R}^{2n})$, i.e. z is a classical solution of (2.3). Therefore $\bar{H}(z(t)) \equiv$ constant so $\Psi(z) = 1$ implies that $z(t) \in \mathcal{D}$. Lastly since $\lambda \neq 0$, making the change of time scale $t \to \lambda t$ shows z is a $2\pi\lambda$ periodic solution of (HS). The proof is complete.

§5. A Dual Approach

In this section another existence proof will be given for a critical value of $A|_M$. This approach is "dual" to the previous one in the spirit of [19]. The critical value obtained in this section may differ from that given by (4.6).

The new critical value will as in §4 be obtained as a minimax. Let

$$\Lambda = \{g \in C(M,M) \mid g \text{ satisfies properties } 1^\circ\text{-}3^\circ \text{ of } \Gamma$$
$$\text{and } 4^\circ \quad P^-g = \beta^-(z)z^- + B^-(z) \text{ where } \beta^- \in C(M,[\beta_1,1]),$$
$$\beta_1 = \beta_1(g) > 0, \text{ and } P_2B^- \text{ is compact}\}.$$

As in §4, there is an intersection theorem associated with Λ.

Theorem 5.1: If $g \in \Lambda$, $g(M^-) \cap M^+ \neq \emptyset$.

Proof: Set $g_m = P_m g \in C(M^- \cap E_m, E_m)$ where E_m and P_m are as in the proof of Theorem 4.7. By properties 1°-2° of Λ, g_m is equivariant and $g_m(z) = z$ on E^0. Hence Proposition 2 of [17] can again be invoked - this time with f being the orthogonal projector of $E_m^- \oplus E^0 \oplus L^+$ onto $E_m^- \oplus E^0$ composed with g_m - to obtain $z_m \in M^- \cap E_m$ such that $g_m(z_m) \in E_m^+$. By Proposition 3.26, (z_m) is a bounded sequence in $L^2(S^1,\mathbb{R}^{2n})$. Therefore by (4.2), (z_m^+) is bounded in L^2 and therefore in E since $z_m^+ \in L^+$ which is finite dimensional. Since E^0 is L^2 orthogonal to $E^- \oplus E^+$ via the definition of these spaces, (z_m^0) is bounded in E. We claim (z_m^-) is also bounded in E. If not,

$$A(z_m) = \frac{1}{2}(\|z_m^+\|^2 - \|z_m^-\|^2) \to -\infty .$$

But then by property 2° of Λ, $g_m(z_m) = z_m$ for large m so $z_m \in M^- \cap E^+ = M \cap L^+$. This implies $z_m = z_m^+$ for large m, $z_m^- = 0$, and (z_m^-) is a bounded sequence.

Since (z_m) is a bounded sequence, it possesses a subsequence which converges weakly in E to $z \in E$. By property 4° of Λ,

$$P_2P^-g_m(z_m) = 0 = \beta^-(z_m)q_m^- + P_mP_2B^-(z_m) \tag{5.2}$$

and $P_mP_2B^-(z_m)$ has a convergent subsequence in $W^{1,2}(S^1,\mathbb{R}^n)$. Hence so does q_m^-.

Therefore $p_m^- = D_{qm}^-$ does also in $L^2(S^1, R^n)$. Since $E^0 \oplus L^+$ is finite dimensional, it follows that $z_m \to z$ in E and $z \in M^-$. Since g is continuous, $g_m(z_m) \to g(z) \in M \cap E^+ = M^+$.

Now define

$$\tilde{c} = \inf_{g \in \Lambda} \sup_{w \in g(M^-)} A(w) . \tag{5.3}$$

<u>Theorem 5.4:</u> $\tilde{c}$ is a critical value of $A|_M$ with $\underline{\alpha} \leq \tilde{c} \leq \overline{\alpha}$.

<u>Proof:</u> Since $\mathrm{id} \in \Lambda$, $\tilde{c} \leq \overline{\alpha}$. Moreover by Theorem 5.1, if $g \in \Lambda$, $g(M^-) \cap M^+ \neq \emptyset$. Therefore

$$\tilde{c} \geq \inf_{M^+} A \equiv \underline{\alpha} .$$

Finally using Remark 3.54, the proof that $\tilde{c}$ is a critical value of $A|_M$ follows the same lines as the proof of Theorem 4.10 and we will omit it.

§6. An a Priori Bound for the Period

Theorem 1 establishes the existence of a periodic solution of (HS) as a critical point of $A|_M$. In this section, in a somewhat more general setting, an a priori bound will be obtained for the period of any periodic solution of (HS) in terms of $A(z)$ and various constants determined from (H_1)-(H_3). Writing (HS) in the form (2.3), the period is $2\pi\lambda$; hence our a priori bound is for λ.

Theorem 6.1: Suppose H satisfies (H_1)-(H_3) and $z \in C^1(S^1, R^{2n})$ is a solution of (2.3) with $\lambda \neq 0$. Then there are constants $\bar{a} > \underline{a} > 0$ independent of z such that

$$\underline{a}|A(z)| \leq |\lambda| \leq \bar{a}|A(z)| . \tag{6.2}$$

Proof: Without loss of generality we can assume λ and $A(z)$ are positive. Writing (2.3) as

$$\dot{p} = -\lambda H_q(z) \tag{6.3}$$

$$\dot{q} = \lambda H_p(z) , \tag{6.4}$$

equation (6.4) implies

$$A(z) = \lambda \int_0^{2\pi} p \cdot H_p(z)dt . \tag{6.5}$$

Consequently

$$A(z) \leq 2\pi\lambda \max_{(\xi,\eta)=\zeta\in\mathcal{D}} \xi \cdot H_p(\zeta) \tag{6.6}$$

and this gives the lower bound for λ in (6.2).

Next from (6.3)

$$-\lambda \int_0^{2\pi} |H_q|^2 dt = \int_0^{2\pi} \dot{p} \cdot H_q dt = -\int_0^{2\pi} p \cdot (H_{qq}\dot{q} + H_{qp}\dot{p})dt$$

$$= -\lambda \int_0^{2\pi} p \cdot (H_{qq}H_p - H_{qp}H_q)dt$$

or

$$(6.7) \qquad 0 = \lambda \int_0^{2\pi} [|H_q|^2 + (p \cdot (H_{qp}H_q - H_{qq}H_p))]dt \ .$$

Adding b times (6.7) to (6.5) gives

$$(6.8) \qquad A(z) = \lambda \int_0^{2\pi} [p \cdot H_p + b|H_q|^2 + (bp \cdot (H_{qp}H_q - H_{qq}H_p))]dt \ .$$

By (H_2), there is a $\gamma > 0$ such that

$$|H_q(0,q)| \geq \gamma \quad \text{if} \quad (0,q) \in \mathcal{D} \ .$$

Therefore there is a $\sigma > 0$ such that

$$(6.9) \qquad |H_q(p,q)| \geq \frac{\gamma}{2} \quad \text{if} \quad (p,q) \in \mathcal{D} \quad \text{and} \quad |p| \leq \sigma \ .$$

Making σ still smaller if necessary, it can be assumed that

$$(6.10) \qquad |p \cdot (H_{qp}H_q - H_{qq}H_p)| \leq \frac{\gamma^2}{8} \quad \text{if} \quad (p,q) \in \mathcal{D} \quad \text{and} \quad |p| \leq \sigma \ .$$

Writing (6.8) as

$$(6.11) \qquad \frac{A(z)}{\lambda} \equiv I_1 + I_2$$

where I_1 denotes the integral of the right hand side of (6.8) over $\{t \in [0,2\pi] \,|\, |p(t)| \leq \sigma\}$ and I_2 denotes the complementary integral, lower bounds will be obtained for I_1, I_2. By (6.9)-(6.11), if

$$\ell \equiv \text{meas}\{t \in [0,2\pi] \,|\, |p(t)| \leq \sigma\} \ ,$$

then

$$(6.12) \qquad I_1 \geq b\left(\frac{\gamma^2}{4} - \frac{\gamma^2}{8}\right)\ell = b\,\frac{\gamma^2}{8}\,\ell \ .$$

To estimate I_2, let

$$M_1 \equiv \max_{z \in \mathcal{D}} |p \cdot (H_{qp}H_q - H_{qq}H_p)|$$

and

$$\omega(\sigma) \equiv \frac{1}{2M_1} \min_{z \in \mathcal{D}, |p| \geq \sigma} p \cdot H_p(z) \ .$$

Then

$$(6.13) \qquad I_2 \geq (2\omega(\sigma) - b)M_1(2\pi - \ell) .$$

Choosing $b = \omega(\sigma)$ and combining (6.11)-(6.13) yields

$$(6.14) \qquad \frac{A(z)}{\lambda} \geq \omega(\sigma)\left(\frac{\gamma^2}{8}\ell + M_1(2\pi - \ell)\right)$$

$$\geq 2\pi\omega(\sigma)\min\left(\frac{\gamma^2}{8}, M_1\right) = \kappa(\sigma) .$$

Thus the upper bound for λ in (6.2) holds with $\bar{a} = \kappa(\sigma)^{-1}$.

Remark 6.15: The constant $\underline{a}$ in (6.2) depends only on C^1 bounds for H on $\mathcal{D}$ while $\bar{a}$ depends on C^2 bounds for H. To obtain an existence theorem for (HS) when H is merely in C^1, a better estimate for $\bar{a}$ is needed. Let $W(z) \in \mathbb{R}^n$ such that W is C^1 in a neighborhood of $\mathcal{D}$. Then as in (6.7) we get

$$(6.16) \qquad 0 = \lambda \int_0^{2\pi} [(W(z) \cdot H_q) + (p \cdot (W_p(z)H_q - W_q(z)H_p))]dt .$$

Suppose W satisfies

$$(6.17) \qquad W(z) \cdot H_q(z) \geq \frac{\gamma^2}{2} \quad \text{if} \quad z = (0,q) \in \mathcal{D} .$$

Arguing as in (6.8)-(6.15) then yields

$$(6.18) \qquad \frac{A(z)}{\lambda} \geq \omega(\sigma)\left(\frac{\gamma^2}{8}\ell + \bar{M}_1(2\pi - \ell)\right) \equiv \bar{\kappa}(\sigma)$$

where

$$\bar{M}_1 \equiv \max_{z\in\mathcal{D}} |p \cdot (W_pH_q - W_qH_p)| .$$

Therefore we get an upper bound for λ of the desired type.

The existence of a W as in (6.17) follows from a result of Palais [20]. If E is a real Banach space, $\mathcal{O} \subset E$, and $\Phi \in C^1(\mathcal{O},\mathbb{R})$, then $w \in E$ is a pseudogradient vector for Φ at $z \in \mathcal{O}$ if

$$(6.19)\qquad \text{(i)}\quad \|w\| \le 2\|\Phi'(z)\|$$

$$\text{(ii)}\quad \langle \Phi'(z), w\rangle_{E^*,E} \ge \|\Phi'(z)\|^2 .$$

If $\Phi \in C^1(E,\mathbf{R})$, $\tilde{E} = \{z \in E \mid \Phi'(z) \neq 0\}$, $W(z)$ is locally Lipschitz continuous, and $W(z)$ is a pseudogradient vector for all $z \in \tilde{E}$, $W(z)$ is called a pseudogradient vector field on $\tilde{E}$. Palais has proved [20].

<u>Lemma 6.20</u>: If $\Phi \in C^1(E,\mathbf{R})$, there exists a pseudogradient vector field for W on $\tilde{E}$.

Choosing $E \equiv \mathbf{R}^{2n}$ and using (H_1)-(H_2), it is easy to verify there exists such a W in our setting. Moreover by using a smooth partition of unity in the proof of Lemma 6.20 - see e.g. Lemma 1.6 of [15] - it can be assumed that W is smooth. Thus the estimate (6.2) holds even when $H \in C^1$.

§7. A More Refined Existence Theorem

The goal of this section is to prove

Theorem 7.1: Suppose H satisfies

(H_1') $$H \in C^1(\mathbb{R}^{2n},\mathbb{R})$$

and (H_2)-(H_3). Then (HS) has a periodic solution on $\mathcal{D} \equiv H^{-1}(1)$.

Proof: Since H satisfies (H_1'), one can find a sequence of functions $H_m \in C^2(\mathbb{R}^{2n},\mathbb{R})$ such that H_m satisfies (H_2)-(H_3) and H_m converges to H in the C^1 norm uniformly on compact subsets of $\mathbb{R}^{2n}$. By Theorem 1, the equation

(7.2) $$\dot{z}_m = \lambda_m J H_{mz}(z_m)$$

has a 2π periodic solution z_m lying on $\mathcal{D}_m \equiv H_m^{-1}(1)$ with λ_m satisfying (6.2) with constants $\underline{a}_m$, $\bar{a}_m$. Moreover

(7.3) $$\underline{\alpha}_m \equiv \inf_{z\in M_m^-} A(z) \le A(z_m) \le \sup_{z\in M_m^+} A(z) \equiv \bar{\alpha}_m \ .$$

We claim there exist constants $\alpha^* > \alpha_* > 0$ such that

(7.4) $$\alpha_* \le \underline{\alpha}_m \le \bar{\alpha}_m \le \alpha^*$$

for all $m \in \mathbb{N}$ and constants $a^* > a_* > 0$ such that

(7.5) $$a_* A(z_m) \le \lambda_m \le a^* A(z_m)$$

for all large $m \in \mathbb{N}$. Assuming (7.4)-(7.5) for the moment, it follows that the sequence (λ_m) is bounded away from 0 and ∞. Since $\mathcal{D}_m$ is near $\mathcal{D}$ for all large m, the functions z_m are bounded in $L^\infty(S^1,\mathbb{R}^{2n})$. Hence by (7.2), (z_m) is bounded in $C^1(S^1,\mathbb{R}^{2n})$. The Arzela-Ascoli Theorem and (7.2) then imply that (λ_m,z_m) converges in $\mathbb{R}\times C^1(S^1,\mathbb{R}^{2n})$ to (λ,z) satisfying

$$\dot{z} = \lambda J H_z(z) \ .$$

To complete the proof of Theorem 7.1, (7.4)-(7.5) must be verified. For the latter inequalities, the constant $\underline{a}_m$ is determined from (6.6) with H replaced by H_m. Since $H_m \to H$ in C^1 uniformly in a neighborhood of $\mathcal{D}$, an a_* which works for all large $m \in \mathbb{N}$ can be determined. The same reasoning, together with the proofs of Theorem 6.1 and Remark 6.15 supply an a^* independent of m provided that there exists a $W(z)$

satisfying (6.17) with $H = H_m$ but γ and W independent of m. Since a γ exists such that

$$|H_q(z)| > \gamma^2 \quad \text{for} \quad z = (0,q) \in \mathcal{D}, \tag{7.6}$$

the convergence of H_m to H implies there is a $k \in \mathbb{N}$ such that

$$H_{mq}(z) \cdot H_{\ell q}(z) > \frac{\gamma^2}{2} \tag{7.7}$$

for $z = (0,q) \in \mathcal{D}_m$ and for all $m, \ell > k$. Therefore W can be taken to be $H_{kq}(z)$.

Lastly to check (6.4), first note that by the construction of $\bar{H}$ in §1, it can be assumed that there are constants r_1, r_2 such that $\bar{H}_m(z) > r_1|z|^2 - r_2$ for all $z \in \mathbb{R}^{2n}$ (independently of m). Therefore $z \in M_m \equiv \Psi_m^{-1}(1)$ implies that $\|z\|_{L^2} < 2\pi(1 + r_2)r_1^{-1}$. The proof of Lemma 4.3 then shows how to obtain α^*. To get α_*, we argue indirectly. If there were no such constant, then for each $m \in \mathbb{N}$, there is a $\zeta_m \in M_m^+$ and such that $\|\zeta_m\| \to 0$ as $m \to \infty$. Suppose $\zeta_m = (\xi_m, \eta_m)$. Then $\eta_m \to 0$ in $W^{1,2}(S^1, \mathbb{R}^n)$ and a fortiori $\eta_m \to 0$ in $L^\infty(S^1, \mathbb{R}^n)$ while $\xi_m \to 0$ in $L^2(S^1, \mathbb{R}^n)$. Since $V_m(q) = U_m(q) = \rho^{-1}|q|^2$ for small q independently of m via the definition of $\bar{H}$ in §1, $_m(\eta_m) \to 0$ as $m \to \infty$. By (K_4) of Proposition 1.3,

$$K_m(z) < r(1 + |z|)|p|$$

where r is independent of m. Consequently $K_m(\zeta_m) \to 0$ as $m \to \infty$. But then $1 = \Psi_m(\zeta_m) \to 0$ as $m \to \infty$, a contradiction. The proof is complete.

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The main result of this paper is the following theorem: Let $p,q \in \mathbf{R}^n$, $H = H(p,q) \in C^2(\mathbf{R}^{2n},\mathbf{R})$ and let $H^{-1}(1)$ be the boundary of a compact neighborhood of 0 with $\nabla H \neq 0$ on $H^{-1}(1)$. If further $p \cdot H_p > 0$ on $H^{-1}(1)$ when $p \neq 0$, then the Hamiltonian system of ordinary differential equations

$$\dot{p} = -H_q(p,q), \quad \dot{q} = H_p(p,q)$$

possesses a periodic solution on $H^{-1}(1)$. The proof involves minimax arguments from the calculus of variations.

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